

A GENERALIZED FLOOR BOUND FOR THE MINIMUM DISTANCE OF GEOMETRIC GOPPA CODES AND ITS APPLICATION TO TWO-POINT CODES

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ABSTRACT. We prove a new bound for the minimum distance of geometric Goppa codes that generalizes two previous improved bounds. We include examples of the bound to one and two point codes over both the Suzuki and Hermitian curves.

1. INTRODUCTION

In [4], Goppa gives a construction of a family of error-correcting codes using two divisors on a curve. He also gives a distance bound based on the degrees of the divisors. Yang et. al. in [14] and Chen et. al. in [1] compute the actual minimum distance of the one-point Hermitian and certain one-point Suzuki codes respectively, often showing that the actual distance is significantly higher than Goppa's designed distance. In [2], Feng and Rao present a decoding algorithm for Goppa codes, which always decodes at least up to half the designed distance. The new distance bound obtained from their algorithm is in general very good as few codes have minimum distance exceeding the Feng-Rao (F-R) distance.

Other efforts to improve and generalize the distance bounds have had some success. In [8], Kirfel and Pelikaan generalize a result of Garcia et al. in [3] and improve the designed distance by taking advantage of pairs of large gaps in the Weierstrass gap sequence at a point P . In [10], Maharaj et al. improve the minimum distance by using the notion of the floor of a divisor to capitalize on one large gap in a multi-point code. We prove a new bound, which generalizes the previous two. It can use more than one gap in the gap sequence, but also applies nicely to multi-point codes. We state our main result below.

Theorem 3.3 (Asymmetric Floor Bound). *Let \mathcal{X} be a curve over K/\mathbb{F}_q of genus g . Let P_1, \dots, P_n be distinct rational points on \mathcal{X} . Define $D := P_1 + \dots + P_n$. Let A , B , and Z be divisors with support outside of D such that Z is effective, $\ell(A) = \ell(A - Z)$, and $\ell(B) = \ell(B + Z)$. Then, putting $G = A + B$ yields*

$$d(C_\Omega(D, G)) \geq \deg(G) - (2g - 2) + \deg(Z).$$

For some codes, the asymmetric bound achieves parameters that were unattainable by the previous bounds. See Section 4 for specific examples. Section 2 gives definitions and notation that will be used throughout the paper. Section 3 gives a

formal statement of all three mentioned bounds, a proof of the asymmetric bound, and relationships among the three.

2. PRELIMINARIES

Unless otherwise noted, we follow the same definitions and notations as in [10]. If P is a rational point on a curve \mathcal{X} that is defined over \mathbb{F}_q with function field K , then v_P represents the discrete valuation corresponding to P . For a divisor A , we denote the support of A as $\text{Supp}(A)$. If B is another divisor, then we define the greatest common divisor of A and B by

$$\gcd(A, B) := \sum_P \min\{v_P(A), v_P(B)\}P.$$

From A , we can create two vector spaces, one of rational functions from the function field K , and one of rational differentials:

$$\mathcal{L}(A) := \{f \in \mathbb{F} : (f) + A \succcurlyeq 0\} \cup \{0\}$$

and

$$\Omega(A) := \{\eta \in \Omega : (\eta) \succcurlyeq A\} \cup \{0\}.$$

We denote the dimension of $\mathcal{L}(A)$ over \mathbb{F}_q by $\ell(A)$ and the dimension of $\Omega(A)$ over \mathbb{F}_q by $i(A)$.

Using the distinct rational points $P_1, \dots, P_n, Q_1, \dots, Q_m$ on \mathcal{X} to form the two divisors

$$D := P_1 + \dots + P_n \text{ and } G := \sum_i \alpha_i Q_i \quad \alpha_i \in \mathbb{Z},$$

we can define two linear m -point codes:

$$C_{\mathcal{L}}(D, G) := \{(f(P_1), \dots, f(P_n)) : f \in \mathcal{L}(G)\}$$

and

$$C_{\Omega}(D, G) := \{(\text{res}_{P_1}(\eta), \dots, \text{res}_{P_n}(\eta)) : \eta \in \Omega(G - D)\}.$$

Both of these codes have length n . The dimension of $C_{\mathcal{L}}$ is $\ell(G) - \ell(G - D)$ and the designed distance is $n - \deg(G)$. The dimension of C_{Ω} is $i(G - D) - i(G)$ and the designed distance is $\deg(G) - (2g - 2)$.

The floor of A is defined in [10] as the unique divisor, A' , of minimum degree such that $\mathcal{L}(A') = \mathcal{L}(A)$ and is denoted $\lfloor A \rfloor$.

Finally, we say that an integer α is an A -gap at a point P if and only if $\mathcal{L}(A + \alpha P) = \mathcal{L}(A + (\alpha - 1)P)$.

3. ASYMMETRIC BOUND

We start by stating two theorems. The first appears as Theorem 2.10 in [10], the second as Proposition 3.10 in [8].

Theorem 3.1 (Floor Bound). *Let K/\mathbb{F}_q be a function field of genus g . Let $D := P_1 + \dots + P_n$ where P_1, \dots, P_n are distinct rational places of F , and let $G := H + \lfloor H \rfloor$ be a divisor of F such that the support of H does not contain any of the*

places P_1, \dots, P_n . Set $E_H := H - \lfloor H \rfloor$. Then $C_\Omega(D, G)$ is an $[n, k, d]$ code whose parameters satisfy

$$d \geq \deg(G) - (2g - 2) + \deg(E_H) = 2\deg(H) - (2g - 2).$$

Theorem 3.2 (K-P Bound). *Suppose that each of the integers $\alpha, \alpha + 1, \dots, \alpha + t$ is an F -gap at P and $\beta - t, \dots, \beta - 1, \beta$ are G -gaps at P . Put $H = F + G + (\alpha + \beta - 1)P$. Suppose $D := P_1 + \dots + P_n$, where the P_i are n distinct rational points, each not equal to P and not belonging to the support of H . Then the minimum distance of $C_\Omega(D, H)$ is at least $\deg(H) - (2g - 2) + (t + 1)$.*

We adapt the proof of the floor bound to prove the following theorem, which generalizes both the floor bound and the K-P bound.

Theorem 3.3 (Asymmetric Floor Bound (AF Bound)). *Let \mathcal{X} be a curve over K/\mathbb{F}_q of genus g . Let P_1, \dots, P_n be distinct rational points on \mathcal{X} . Define $D := P_1 + \dots + P_n$. Let A, B, G , and Z be divisors with support outside of D such that Z is effective, $\ell(A) = \ell(A - Z)$, $\ell(B) = \ell(B + Z)$, and $G = A + B$. Then*

$$d(C_\Omega(D, G)) \geq \deg(G) - (2g - 2) + \deg(Z).$$

Proof. Let $\eta \in \Omega(G - D)$ such that the code word $\vec{c} := (\text{res}_{P_1}(\eta), \dots, \text{res}_{P_n}(\eta))$ is of minimum nonzero weight. Without loss of generality, we may assume that $c_i \neq 0$ for $1 \leq i \leq d$ and $c_i = 0$ for $d < i \leq n$. Let $D' := P_1 + \dots + P_d$. Since \vec{c} is zero outside D' , we must have that $(\eta) \geq G - D'$. Thus, there exists an effective divisor, E , with support disjoint from that of D' so that $W := (\eta) = G - D' + E$. Taking degrees of both sides we get that

$$2g - 2 = \deg(G) - d + \deg(E) \Rightarrow d = \deg(G) - (2g - 2) + \deg(E).$$

Observe that,

$$\deg(E) \geq \ell(A + E) - \ell(A) = \ell(A + E) - \ell(A - Z) \geq \ell(A + E) - \ell(A + E - Z).$$

By applying the Riemann-Roch Theorem twice, we get that

$$\ell(A + E) = \deg(A + E) + 1 - g + \ell(W - (A + E))$$

and

$$\ell(A + E - Z) = \deg(A + E - Z) + 1 - g + \ell(W - (A + E - Z))$$

and thus,

$$\ell(A + E) - \ell(A + E - Z) = \deg(Z) + \ell(B - D') - \ell(B + Z - D').$$

Now

$$\mathcal{L}(B + Z - D') \subseteq \mathcal{L}(B + Z) = \mathcal{L}(B)$$

implies that

$$\mathcal{L}(B + Z - D') = \mathcal{L}(B + Z - D') \cap \mathcal{L}(B) = \mathcal{L}(\gcd(B + Z - D', B)).$$

Since B and Z have support outside of D' and since Z is effective, $\gcd(B + Z - D', B) = B - D'$, and thus $\ell(B + Z - D') = \ell(B - D')$.

Finally we see that this gives that $\deg(E) \geq \deg(Z)$.

□

We note that the above proof assumes that the code $C_\Omega(D, G)$ is nontrivial and thus has a codeword of nonzero weight. If the code is trivial, the question of minimum distance is not very interesting.

Remark 3.4. Taking the special case $A = H$, $B = \lfloor H \rfloor$ and $Z = H - \lfloor H \rfloor$ in the AF bound gives the floor bound. Unlike the floor bound, the AF bound can always be applied. For an arbitrary divisor G it is not always the case that there exists a divisor H such that G can be written as the sum of H and $\lfloor H \rfloor$; however one can always let $A = G$ and $B = Z = 0$ and use the asymmetric bound trivially. Beyond this, there are many examples (see §4) where the floor bound cannot be applied, but the asymmetric bound gives an improvement over the designed distance. Additionally, there are examples where the floor bound does apply, but the more general choice of A and B gives a greater improvement.

Remark 3.5. Setting $A = F + (\alpha + t)P$, $B = G + (\beta - t - 1)P$ and $Z = (t + 1)P$ in the AF bound yields the K-P bound. So in the case of one point codes, the AF bound reduces to the K-P bound. Note that in its statement, however, Theorem 3.2 makes no assumption about the support of F or G containing only P . Thus, the bound can be applied to n -point codes, but only takes advantage of “one-dimensional” gaps, while the asymmetric bound uses “multi-dimensional” gaps. We also note that in order to prove their bound, Kirfel and Pellikaan compare their estimate to the F-R bound and show that their estimate is always lower. This shows that in the one point case, the asymmetric bound will not improve on Feng-Rao.

4. EXAMPLES

In this section we give several examples of how the asymmetric bound improves over the floor bound and the K-P bound.

Example 4.1. Consider the one point codes generated by the Suzuki curve, $y^8 - y = x^{10} - x^3$ over $\overline{\mathbb{F}}_8$. This is a curve of genus $g = 14$. If we let $G = 41P_\infty$ and let D be the sum of all the other rational points on the curve, then we get that $C_\Omega(D, G)$ has designed distance 15. From Table 2 in [1] and the equivalence $C_\Omega(D, aP_\infty) = C_\mathcal{L}(D, (90 - a)P_\infty)$, we know that the actual distance is 16. The Weierstrass semi-group is generated by 8, 10, 12, and 13, so we see that there is no way to write G as $H + \lfloor H \rfloor$, so that the floor bound does not apply. However, if we let $A = 27P_\infty$, $B = 14P_\infty$, and $Z = P_\infty$ then Theorem 3.3 applies and we get that the minimum distance is at least $15 + \deg(Z) = 16$, meeting the actual distance. Also, note that the K-P bound predicts this as well since we are dealing with a one point code.

Example 4.2. This example illustrates the idea that the asymmetric bound can exploit multiple “multi-dimensional” gaps to yield larger distance estimates. Consider the two point code, $C_\Omega(D, G)$, generated by the same curve as in Example 4.1 where $G = 32P_{0,0} + 1P_\infty$, and D is the sum of the remaining rational points. The designed distance of this code is seven. By using knowledge of the Weierstrass semi-group of the pair $(P_{0,0}, P_\infty)$ computed in [11] we can apply all three bounds from the previous section. By letting $H = 16P_{0,0} + 1P_\infty$, we can apply the floor bound to get minimum distance at least eight. Letting $S = 1P_{0,0} + 0P_\infty$, $T = 0P_{0,0} + 1P_\infty$, $\alpha = 18$, $\beta = 14$, and $t = 1$ shows that the K-P bound also gives minimum distance at least eight. However, using the asymmetric bound with $A = 14P_{0,0} + 1P_\infty$ and

$B = 18P_{0,0} + 0P_\infty$, we get $Z = 1P_{0,0} + 1P_\infty$ and a minimum distance of at least nine (See Table 2).

Example 4.3. Consider the two point code, $C_\Omega(D, G)$, generated by the same curve as in Example 4.1 where $G = 15P_{0,0} + 17P_\infty$, and D is the sum of the remaining rational points. If we let $A = 2P_{0,0} + 14P_\infty$ and $B = 13P_{0,0} + 3P_\infty$, then $Z = 2P_{0,0} + 1P_\infty$ and the asymmetric bound gives distance at least nine while $C_\Omega(D, G)$ has designed distance of six (See Table 2). The floor bound, however, does not apply to this code or any linearly equivalent code.

In the following we show the improvement over the designed distance achieved by the AF bound for two particular families of two-point codes; first the Hermitian codes over \mathbb{F}_{16} , then the Suzuki codes over \mathbb{F}_8 .

	0	1	2	3	4		0	1	2	3	4		0	1	2	3	4	
6					+						*						2	6
7				+	+					*	*						3	7
8			+	+	+				2	*	*					3	2	8
9		+	+	+			*	*	*					2	1	1		9
10		+	+	+			1	2	*					1	2	1		10
11	+	+	+	+	+		1	2	3	*	*			1	2	3	2	11
12	+	+	+	+	+		2	3	*	*	*			2	3	2	1	12
13	+	+	+				*	*	*					1	2	1		13
14		+	+				*	*						1	1			14
15		+	+				*	*						1	1			15
16	+	+	+	+	+		*	1	*	*	*			1	1	1	1	16
17	+	+					*	*						1	1			17
18		+					1							1				18
19		+					*							1				19
20		+					1							1				20
21	+						1							1				21
	a						b						c					

Table 1 - Hermitian 2-point codes over \mathbb{F}_{16}

- Codes where the actual distance is greater than the designed distance.
- Improvement that the floor bound gives over the designed distance.
- Improvement that the asymmetric bound gives over the designed distance.

Let \mathcal{X} be the Hermitian curve $y^4 + y = x^5$ over \mathbb{F}_{16} , then $g(\mathcal{X}) = 6$. In Table 1a, a plus “+” designates those coordinates (x, y) such that the $C_\Omega(D, G)$ code with $G = xP_{0,0} + yP_\infty$, D the sum of the remaining rational points, and $\deg(G) \geq 10$ has minimum distance strictly greater than the designed distance. The coordinates (x, y) were calculated using Magma. Any effective G can be found in this table by use of the linear equivalence $5P_{0,0} \sim 5P_\infty$. In Tables 1b and 1c, the number n in position (x, y) gives the improvement over the designed distance predicted by the floor and asymmetric bounds, respectively. An asterisk “*” in Table 1b indicates that the floor bound either gave no improvement, or could not be applied. Note that Theorem 3.3 gives an improvement for every code in Table 1a whose actual distance is greater than its designed distance. Also note that more can be concluded from

the information in Table 1b by considering containment of codes. For example, the code with $G = 12P_{0,0} + 2P_\infty$ is contained in the code with $G = 12P_{0,0} + 1P_\infty$, thus its minimum distance is at least that of the larger code.

	0	1	2	3	4	5	6	7	8	9	10	11	12
14													2
15												3	3
16											3	2	3
17										3	3	2	3
18									3	3	3	2	2
19								4	3	3	3	2	2
20							4	3	3	3	2	2	1
21						3	3	3	2	2	1	1	
22					3	3	3	3	2	2	1	1	1
23				3	3	3	3	2	1	1		1	
24			3	2	2	2	2	2	1	1	1	1	1
25		2	3	3	3	2	2	1		1		1	
26		1	2	2	2	2	2	1		1		1	
27	1	2	3	2	2	2	2	2	1	2	1	2	1
28	2	3	4	3	3	3	2	3	2	2	2	1	1
29	1	2	3	2	2	2	1	2	1	1	1		
30	1	2	3	2	2	2	2	2	1	1	1	1	1
31	1	2	3	2	2	2	1	1	1				
32	1	2	2	1	2	1	2	1	1	1	1	1	1
33	1	2	3	2	2	1	1						
34		1	2	1	1	1	1						
35	1	2	2	1	1		1						
36		1	2	1	1		1						
37	1	2	1		1		1						
38		1	1		1		1						
39		1	1		1		1						
40	1	1	1	1	1	1	1	1	1	1	1	1	1
41	1	1	1	1	1	1							
42		1	1		1								
43	1	1	1	1									
44		1	1										
45	1	1											
46		1											
47		1											
48		1											
49		1											
50		1											
51		1											
52		1											
53	1												

Table 2

The asymmetric bound improvement for the Suzuki curve over \mathbb{F}_8 .

Again consider the two-point Suzuki codes over \mathbb{F}_8 as in Examples 4.2 and 4.3. Table 2 gives the improvement over the designed distance of these codes using Theorem 3.3. Table 2 is the analog for the Suzuki codes of Table 1c. We list those divisors G such that $\deg G \geq 2g - 2 = 26$ and such that Theorem 3.3 predicts an improvement over the designed distance.

Note. We note that Tables 1 and 2 were computed using information about the Weierstrass semigroup on two points that appeared in [12] and [11] respectively.

We also note that the AF bound does not reach the actual distance in all cases. Nor does it improve on the F-R bound for one-point codes. In the general m -point case, both the floor bound and the AF bound produce improvements for the minimum distance yet they lack decoding algorithms to exploit those improvements.

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